Greener Journal of Physics and Natural Sciences
Vol. 4(1), pp. 1-6, 2019
ISSN: 2384-6410

DOI Link: https://doi.org/10.15580/GJPNS.2019.1.102619193
https://gjournals.org/GJPNS

Condensed Lagrange Equations in Theoretical Mechanics

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ARTICLE INFO

Article No.:102619193
Type: *Review*
DOI:10.15580/GJPNS.2019.1.102619193

Submitted: 26/10/2019
Accepted: 29/10/2019
Published: 22/11/2019

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Keywords: rigid body mechanics; condensed Lagrange equations; absolute general motion

ABSTRACT

In this article, new equations called Condensed Lagrange equations (CLE) are defined. With their help, the rigid body absolute general motion is studied. The rigid body is considered to be homogeneous and unsymmetrical. CLEs are similar in structure to the classical Lagrange equations from second type, applied in vector-matrix form, but CLE are differing from them by four indicators. These differences are commented in detail in this article. The use of the CLE is fully equivalent to the application of the theorem, called Theorem for change of the rigid body generalized impulse. Using CLEs the differential equations in a matrix form, describing the rigid body absolute general motion, are obtained. CLEs enrich the theory of Rigid Body Mechanics. Moreover, CLEs represent a second alternative variant of the Theorem for change of the rigid body generalized impulse, they serve for verification, and finally, they make the study completely.

1. INTRODUCTION

On April 5th, 1788, the great Italian-French mathematician and physicist Joseph Louis Lagrange (1736-1813) has presented to the ParisAcademy his famous book *M'ecaniqueanalitique*. In this book, for the first time, a classical form of differential equations for the study of non-free mechanical systems with many degrees of freedom is formulated. These equations are known as Lagrange equations from the second type, for example (Suslov, 1976; Pars, 1964; Zlatev, 1965; Eiserman, 1974).

Analytical mechanics has been developed over the years. There are many modifications to these equations, for example (Tzenov, 1953; Dolapchiev, 1969).

The Lagrange equations from the second type are scalar. After the introduction of matrices and matrix calculations in1858 by Arthur Cayley (1821-1895), who could reasonably be considered not only a great British but also a world mathematician, it becomes possible to type them in a vector-matrix form, (Cayley, 1858).

Usually, the differential equations of the rigid body generalized motion in matrix form are obtained by means of the two main theorems of Classical Mechanics – The Theorem for change of linear momentum and

Theorem for change of angular momentum are always applied, (Wittenburg, 1977; Featherstone, 2008; Awrejcewicz et al., 2012). Of course, the author of this article managed to unite these two theorems into one. This new theorem is called *Theorem for change of the rigid body generalized impulse*.

In this article, in a shortened form, new equations in vector-matrix form, called *Condensed Lagrange equations* (CLE), are presented. With their help, the differential equations of the rigid body general motion are obtained easily and clearly.

CLE are presented a second additional and alternative variant of the Theorem for change of the rigid body generalized impulse (TCRBGI). Moreover, these equations are served for verification of the TCRBGI and they make the study completely.

CLE has already been used in some author's publications, for example (Ivanov, 2017; Ivanov, 2018). Here, the title of this article coincides with their name. In this way, the main purpose of the study is to promote them to a wide range of scientists.

2. RIGID BODY KINETIC ENERGY

The kinetic energy of a free rigid body that performs an absolutely general motion, at an arbitrarily chosen pole O, has the form (Fig1):

$$E_k = \frac{1}{2}.\left\langle \mathbf{v}_{O,A} \quad \boldsymbol{\omega}_A \right\rangle^T . \begin{bmatrix} \mathbf{M} & \mathbf{S}_{C,A}^T \\ \mathbf{S}_{C,A} & \mathbf{J}_{O,A} \end{bmatrix} . \begin{bmatrix} \mathbf{v}_{O,A} \\ \boldsymbol{\omega}_A \end{bmatrix}, \tag{1}$$

$$E_k = \frac{1}{2}.\mathbf{u}_{O,A}^T.\mathbf{A}_{O,A}.\mathbf{u}_{O,A} . \tag{2}$$

It has been proven the following theorem: the doubling value of the kinetic energy of a body, that performs a general motion, is equal to the scalar product of the vector-real generalized velocity and the vector-generalized impulse of this rigid body.

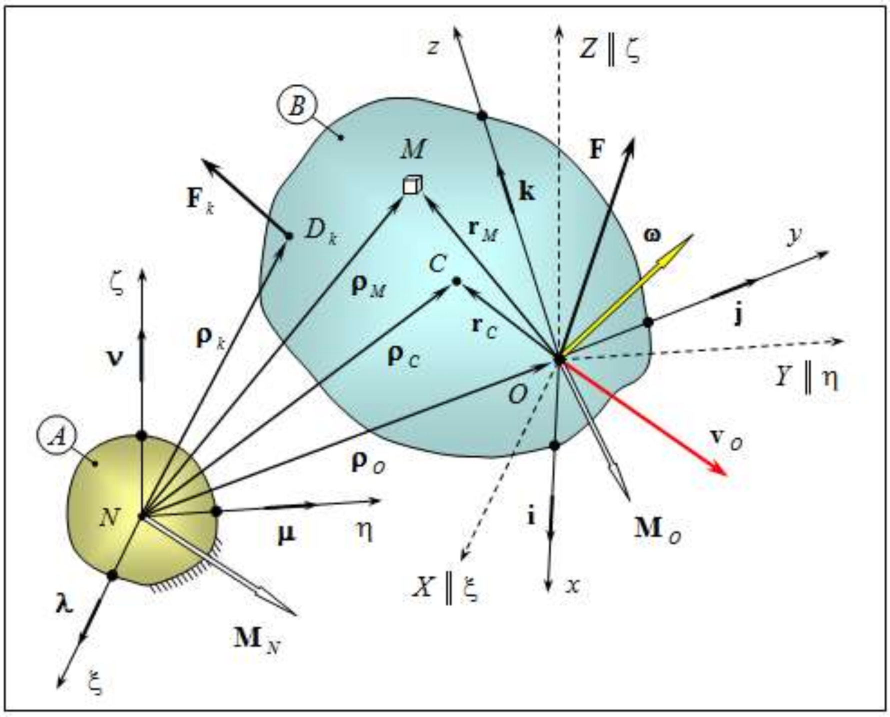

Fig.1: Dynamics of rigid body absolute general motion

Indeed, the lower two equals lead to the same result, since the matrix of mass and inertial characteristics $\mathbf{A}_{O,A}$ is symmetrical, or $\mathbf{A}_{O,A}^T = \mathbf{A}_{O,A}$, (Ivanov, 2018, a, b):

$$\mathbf{u}_{O,A}^T.\mathbf{D}_{O,A} = \mathbf{u}_{O,A}^T.\mathbf{A}_{O,A}.\mathbf{u}_{O,A} = 2.E_k , \tag{3}$$

(4) $$\mathbf{D}^{T}_{O,A}.\mathbf{u}_{O,A}=\mathbf{u}^{T}_{O,A}.\mathbf{A}^{T}_{O,A}.\mathbf{u}_{O,A}=\mathbf{u}^{T}_{O,A}.\mathbf{A}_{O,A}.\mathbf{u}_{O,A}=2.E_k\ .$$

If the pole O coincides with the mass center C, (Fig. 2), the kinetic energy will be determined by the König Theorem (Johann Samuel König, 1712-1757):

(5) $$E_k=\frac{1}{2}.\mathbf{u}^{T}_{C,A}.\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\ ,$$

(6) $$\mathbf{A}_{C,A}=\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_{C,A}\end{bmatrix}.$$

3. CONDENSED LAGRANGE EQUATIONS

At an arbitrarily chosen pole O, the main variant of Condensed Lagrange equations has two forms of presentation:

(7) $$\frac{d}{dt}\left[\frac{\partial\,(E_k+E^{*}_{O})}{\partial\,\mathbf{u}_{O,A}}\right]=\mathbf{Q}_{N,A}\ ,$$

(8) $$\frac{d}{dt}\left[\frac{\partial\,(E_k+E^{*}_{O})}{\partial\,\mathbf{u}_{O,A}}\right]=\mathbf{Q}_{O,A}+\overline{\mathbf{T}}_{O,A}.\mathbf{Q}_{O,A}\ .$$

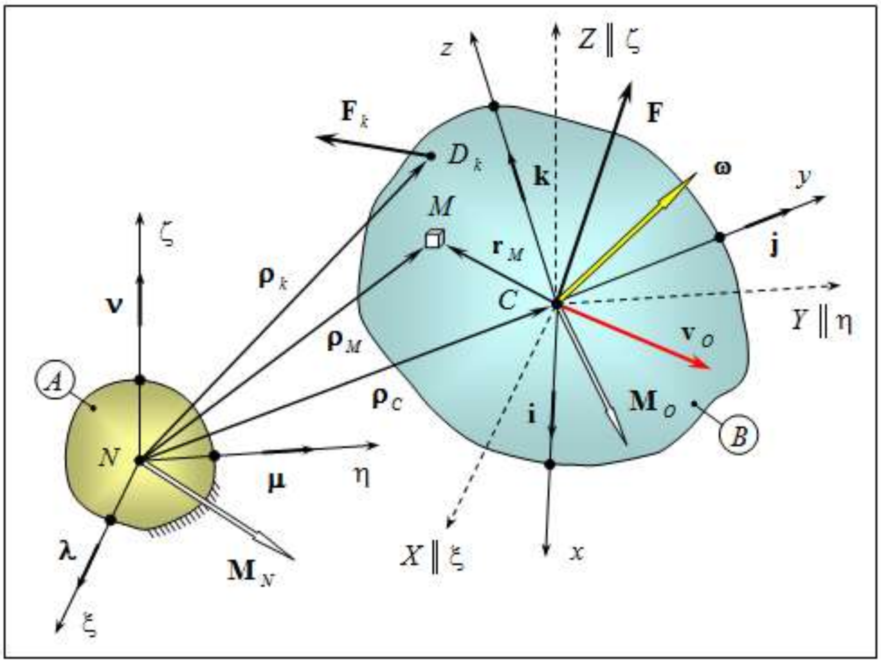

Fig.2: Dynamical model where the pole O coincides with the mass center C

The scalar magnitude E^{*}_{O}, which an additional energy is presented, is constructed by the following manner:

(9) $$E^{*}_{O}=\mathbf{u}^{T}_{O,A}.\widetilde{\mathbf{F}}_{O,A}\ ,$$

(10) $$\widetilde{\mathbf{F}}_{O,A}=\overline{\mathbf{T}}_{O,A}.\mathbf{D}_{C,A}=\overline{\mathbf{T}}_{O,A}.\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\ .$$

Formulas (2) and (9) are substituted in equation (8):

(11) $$\frac{d}{dt}\left(\mathbf{A}_{O,A}.\mathbf{u}_{O,A}\right)+\frac{d\,\widetilde{\mathbf{F}}_{O,A}}{dt}=\mathbf{Q}_{O,A}+\overline{\mathbf{T}}_{O,A}.\mathbf{Q}_{O,A}\ .$$

Then formula (10) is substituted in equation (11):

(12) $$\frac{d}{dt}\left(\mathbf{A}_{O,A}.\mathbf{u}_{O,A}\right)+\frac{d}{dt}\left(\overline{\mathbf{T}}_{O,A}.\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\right)=\mathbf{Q}_{O,A}+\overline{\mathbf{T}}_{O,A}.\mathbf{Q}_{O,A}\ .$$

From now on, in order to obtain the differential equations of the general motion of this asymmetric rigid body at arbitrary chosen pole O, it is necessary to apply the methodology described in the publications, (Ivanov, 2018, a, b).

Using a pole O that coincides with the mass center C of the body, Condensed Lagrange equations will take the following form:

(13) $$\frac{d}{dt}\left[\frac{\partial\,(E_k+E_C^{*})}{\partial\,\mathbf{u}_{C,A}}\right]=\mathbf{Q}_{C,A}+\overline{\mathbf{T}}_{C,A}.\mathbf{Q}_{C,A}\ ,$$

(14) $$E_C^{*}=\mathbf{u}_{C,A}^{T}.\widetilde{\mathbf{F}}_{C,A}\ ,$$

(15) $$\widetilde{\mathbf{F}}_{C,A}=\overline{\mathbf{T}}_{C,A}.\mathbf{D}_{C,A}=\overline{\mathbf{T}}_{C,A}.\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\,.$$

Formulas (5) and (14) are substituted in equation (13):

(16) $$\frac{d}{dt}\left(\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\right)+\frac{d\,\widetilde{\mathbf{F}}_{C,A}}{dt}=\mathbf{Q}_{C,A}+\overline{\mathbf{T}}_{C,A}.\mathbf{Q}_{C,A}\ .$$

Then formula (15) is substituted in equation (16):

(17) $$\frac{d}{dt}\left(\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\right)+\frac{d}{dt}\left(\overline{\mathbf{T}}_{C,A}.\mathbf{A}_{C,A}.\mathbf{u}_{C,A}\right)=\mathbf{Q}_{C,A}+\overline{\mathbf{T}}_{C,A}.\mathbf{Q}_{C,A}\ .$$

Condensed Lagrange equations in formula (13) are fully equivalent to the following kind:

(18) $$\frac{d}{dt}\left[\frac{\partial\,E_k}{\partial\,\mathbf{u}_{C,A}}\right]=\mathbf{Q}_{C,A}\ .$$

The two variants of Condensed Lagrange equations – formulas (13) and (18), lead to the same result.

In order to obtain a more complete perception about Condensed Lagrange equations it is necessary to compare them with the classical Lagrange equations from the second type.

Such a comparison can be made if the classical Lagrange equations from the second type are written in the matrix form as follows:

(19) $$\frac{d}{dt}\left[\frac{\partial\,E_k}{\partial\,\dot{\mathbf{q}}}\right]-\frac{\partial\,E_k}{\partial\,\mathbf{q}}=\mathbf{Q}\ .$$

The main differences between the most general version of Condensed Lagrange equations – formula (8) and the classical Lagrange equations from the second type – formula (19), are as follows.

1. Instead of a private derivative of kinetic energy E_k towards the vector of generalized velocities $\dot{\mathbf{q}}$, which is written in classical Lagrange equations from the second type (19), in Condensed Lagrange equations (8), the analogous private derivative is formed towards the new vector-real generalized velocity $\mathbf{u}_{O,A}$.

2. The private derivative $\dfrac{\partial E_k}{\partial \mathbf{q}}$, which is standing on the second place in the classical Lagrange equations from the second type, in Condensed Lagrange equations such private derivative is absent.

3. Instead of the member $\dfrac{\partial E_k}{\partial \dots}$ in classical Lagrange equations from the second type, in Condensed Lagrange equations, the analog member is modified in the type $\dfrac{\partial (E_k + E_O^*)}{\partial \dots}$, where an additional energy E_O^* is added to the kinetic energy E_k of the body – formulas (9) and (10).

4. Instead of classical form of the vector-generalized forces $\mathbf{Q}$, used in Lagrange equations from the second type, in Condensed Lagrange equations, when the general motion of a rigid body is studied with arbitrary chosen pole O, the new vector-real generalized forces $\mathbf{Q}_{N,A} = \mathbf{Q}_{O,A} + \overline{\mathbf{T}}_{O,A}.\mathbf{Q}_{O,A}$ for the stationary (fixed) center N is defined – formulas (7) and (8).

The name "condensed" is related to the major private derivative $\dfrac{\partial (E_k + E_O^*)}{\partial \mathbf{u}_{O,A}}$, in which differentiation is performed towards the vector-real generalized velocity $\mathbf{u}_{O,A}$.

According to the equation $\mathbf{u}_{O,A} = \mathbf{H}.\dot{\mathbf{q}}$, which is described in many author's articles, (Ivanov, a;Ivanov, 2018, a, b), this vector $\mathbf{u}_{O,A}$ seems to have condensed within itself the other major vector $\dot{\mathbf{q}}$, known as the vector of generalized velocities.

4. CONCLUSION

New equations, called Condensed Lagrange equations, have been formulated. These equations are similar in structure to the classical Lagrange equations from second type, applied in vector-matrix form. Moreover, Condensed Lagrange equations leads to the same result as the Theorem of change the rigid body generalized impulse.

Condensed Lagrange equations differ from the classical Lagrange equations of the second type by four indicators. These indicators are fully described in this article.

The obtained system of nonlinear differential equations in matrix form is convenient for a numerically integrating by the contemporary mathematical programs, which are projected to use matrices and matrix calculations, for example MatLab, MathCAD, Maple, MuPAD and others.

REFERENCES

Awrejcewicz J., Koruba Z., (2012). Classical Mechanics.Applied Mechanics and Mechatronics, Springer.

CayleyA. (1858). A memoir on the theory of matrices. Philosophical Transactions of the Royal Society of London, vol. 148, 17-37.

Dolapchiev B.I., (1969). Summary of the Nilsen-Tzenov equations.Bulletin of the BulgarianAcademy of Sciences.Mathematics, № 10.

Eiserman M.A., (1974). Classical Mechanics.Moscow, Nauka. (in Russian)

Featherstone R., (2008). Rigid Body Dynamics.Algorithms. Springer Science and Business Media, LLC.

Ivanov A.I., (2017). Theoretical Matrix Study of Rigid Body General Motion. Greener Journal of Physics and Natural Sciences, **3** (2), 009-020.

Ivanov A.I., (2017), Theoretical Matrix Study of Rigid Body Pseudo Translational Motion. Greener Journal of Physics and Natural Sciences, **3** (2), 021-031.

Ivanov A.I., (2018). Motion of Asymmetrical Rigid Body in Fluid Area. Annual of the University of Architecture, Civil Engineering and Geodesy, **51** (2), 93-115. (in Bulgarian).

Ivanov A.I., (2018). Motion of Ellipsoid in Fluid Area. Annual of the University of Architecture, Civil Engineering and Geodesy, **51** (2), 117-136. (in Bulgarian).

Pars L.A., (1964). A Treatise on Analytical Dynamics.London, Heineman.

Suslov G.K., (1976). Fundamentals of Analytical Mechanics. Sofia, Nauka. (in Bulgarian)
Tzenov I.A., (1953). On a new form of equations of Analytical Mechanics.Moscow, DAN, USSR, vol. 89, book 1. (in Russian)
Wittenburg J., (1977). Dynamics of Multibody Systems.Springer-Verlag.
Zlatev I.S., (1965). Theoretical Mechanics.Sofia, Nauka i Izkustvo. (in Bulgarian)

Cite this Article: Ivanov A (2019). Condensed Lagrange Equations in Theoretical Mechanics. Greener Journal of Physics and Natural Sciences, 4(1): 1-6, https://doi.org/10.15580/GJPNS.2019.1.102619193.